R.1 Sets

- Basic Definitions
- Operations on Sets

Basic Definitions A **set** is a collection of objects. The objects that belong to a set are called the **elements,** or **members,** of the set. In algebra, the elements of a set are usually numbers. Sets are commonly written using **set braces,** $\{\ \}$. For example, the set containing the elements 1, 2, 3, and 4 is written as follows.

$$\{1, 2, 3, 4\}$$

Since the order in which the elements are listed is not important, this same set can also be written as $\{4, 3, 2, 1\}$ or with any other arrangement of the four numbers.

To show that 4 is an element of the set $\{1, 2, 3, 4\}$, we use the symbol $\in$.

$$4 \in \{1, 2, 3, 4\}$$

Since 5 is *not* an element of this set, we place a slash through the symbol $\in$.

$$5 \notin \{1, 2, 3, 4\}$$

It is customary to name sets with capital letters. If S is used to name the set above, then we write it as follows.

$$S = \{1, 2, 3, 4\}$$

Set S was written by listing its elements. Set S might also be described as

"the set containing the first four counting numbers."

In this example, the notation $\{1, 2, 3, 4\}$, with the elements listed between set braces, is briefer than the verbal description.

The set F, consisting of all fractions between 0 and 1, is an example of an **infinite set,** one that has an unending list of distinct elements. A **finite set** is one that has a limited number of elements. The process of counting its elements comes to an end. Some infinite sets can be described by listing. For example, the set of numbers N used for counting, called the **natural numbers,** or the **counting numbers,** can be written as follows.

$$\boldsymbol{N = \{1, 2, 3, 4, \ldots\}} \quad \text{Natural (counting) numbers}$$

The three dots (*ellipsis points*) show that the list of elements of the set continues according to the established pattern.

Sets are often written using a variable to represent an arbitrary element of the set. For example,

$$\{x \mid x \text{ is a natural number between 2 and 7}\} \quad \text{Set-builder notation}$$

(which is read "the set of all elements x such that x is a natural number between 2 and 7") uses **set-builder notation** to represent the set $\{3, 4, 5, 6\}$. The numbers 2 and 7 are *not* between 2 and 7.

EXAMPLE 1 Using Set Notation and Terminology

Identify each set as *finite* or *infinite*. Then determine whether 10 is an element of the set.

(a) $\{7, 8, 9, \ldots, 14\}$

(b) $\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\}$

(c) $\{x \mid x \text{ is a fraction between 1 and 2}\}$

(d) $\{x \mid x \text{ is a natural number between 9 and 11}\}$

SOLUTION

(a) The set is finite, because the process of counting its elements 7, 8, 9, 10, 11, 12, 13, and 14 comes to an end. The number 10 does belong to the set, and this is written as follows.

$$10 \in \{7, 8, 9, \ldots, 14\}$$

(b) The set is infinite, because the ellipsis points indicate that the pattern continues forever. In this case,

$$10 \notin \left\{1, \tfrac{1}{4}, \tfrac{1}{16}, \tfrac{1}{64}, \ldots\right\}.$$

(c) Between any two distinct natural numbers there are infinitely many fractions, so this set is infinite. The number 10 is not an element.

(d) There is only one natural number between 9 and 11, namely 10. So the set is finite, and 10 is an element.

Now Try Exercises 1, 3, 5, and 7.

EXAMPLE 2 Listing the Elements of a Set

Use set notation, and write the elements belonging to each set.

(a) $\{x \mid x \text{ is a natural number less than } 5\}$

(b) $\{x \mid x \text{ is a natural number greater than 7 and less than } 14\}$

SOLUTION

(a) The natural numbers less than 5 form the set $\{1, 2, 3, 4\}$.

(b) This is the set $\{8, 9, 10, 11, 12, 13\}$.

Now Try Exercise 15.

When we are discussing a particular situation or problem, the **universal set** (whether expressed or implied) contains all the elements included in the discussion. The letter U is used to represent the universal set. The **null set,** or **empty set,** is the set containing no elements. We write the null set by either using the special symbol $\emptyset$, or else writing set braces enclosing no elements, $\{\ \}$.

CAUTION Do not combine these symbols. $\{\emptyset\}$ ***is not the null set.***

Every element of the set $S = \{1, 2, 3, 4\}$ is a natural number. S is an example of a *subset* of the set N of natural numbers, and this is written

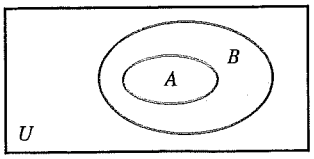
$$S \subseteq N.$$

By definition, set A is a **subset** of set B if every element of set A is also an element of set B. For example, if $A = \{2, 5, 9\}$ and $B = \{2, 3, 5, 6, 9, 10\}$, then $A \subseteq B$. However, there are some elements of B that are not in A, so B is not a subset of A, which is written

$$B \not\subseteq A.$$

By the definition, every set is a subset of itself. Also, by definition, $\emptyset$ is a subset of every set.

If A is any set, then $\emptyset \subseteq A$.



$A \subseteq B$

Figure 1

Figure 1 shows a set A that is a subset of set B. The rectangle in the drawing represents the universal set U. Such diagrams are called **Venn diagrams.**

Two sets A and B are equal whenever $A \subseteq B$ and $B \subseteq A$. Equivalently, $A = B$ if the two sets contain exactly the same elements. For example,

$$\{1, 2, 3\} = \{3, 1, 2\}$$

is true, since both sets contain exactly the same elements. However,

$$\{1, 2, 3\} \neq \{0, 1, 2, 3\},$$

since the set $\{0, 1, 2, 3\}$ contains the element 0, which is not an element of $\{1, 2, 3\}$.

EXAMPLE 1 Examining Subset Relationships

Let $U = \{1, 3, 5, 7, 9, 11, 13\}$, $A = \{1, 3, 5, 7, 9, 11\}$, $B = \{1, 3, 7, 9\}$, $C = \{3, 9, 11\}$, and $D = \{1, 9\}$. Determine whether each statement is *true* or *false*.

(a) $D \subseteq B$ **(b)** $B \subseteq D$ **(c)** $C \nsubseteq A$ **(d)** $U = A$

SOLUTION

(a) All elements of D, namely 1 and 9, are also elements of B, so D is a subset of B, and $D \subseteq B$ is true.

(b) There is at least one element of B (for example, 3) that is not an element of D, so B is *not* a subset of D. Thus, $B \subseteq D$ is false.

(c) C is a subset of A, because every element of C is also an element of A. Thus, $C \subseteq A$ is true, and as a result, $C \nsubseteq A$ is false.

(d) U contains the element 13, but A does not. Therefore, $U = A$ is false.

Now Try Exercises 43, 45, 53, and 55.

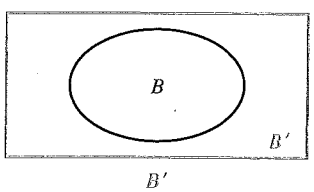


Figure 2

Operations on Sets Given a set A and a universal set U, the set of all elements of U that do not belong to set A is called the **complement** of set A. For example, if set A is the set of all students in your class 30 years old or older, and set U is the set of all students in the class, then the complement of A would be the set of all the students in the class younger than age 30. The complement of set A is written A' (read **"A-prime"**). The Venn diagram in **Figure 2** shows a set B. Its complement, B', is in color.

EXAMPLE 2 Finding the Complement of a Set

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 5, 7\}$, and $B = \{3, 4, 6\}$. Find each set.

(a) A' **(b)** B' **(c)** $\emptyset'$ **(d)** U'

SOLUTION

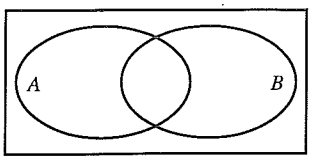
(a) Set A' contains the elements of U that are not in A. Thus, $A' = \{2, 4, 6\}$.

(b) $B' = \{1, 2, 5, 7\}$ **(c)** $\emptyset' = U$ **(d)** $U' = \emptyset$

Now Try Exercise 79.

Given two sets A and B, the set of all elements belonging both to set A ***and*** to set B is called the **intersection** of the two sets, written $A \cap B$. For example, if $A = \{1, 2, 4, 5, 7\}$ and $B = \{2, 4, 5, 7, 9, 11\}$, then we have the following.

$$A \cap B = \{1, 2, 4, 5, 7\} \cap \{2, 4, 5, 7, 9, 11\} = \{2, 4, 5, 7\}$$



$A \cap B$

Figure 3

The Venn diagram in **Figure 3** shows two sets A and B. Their intersection, $A \cap B$, is in color.

Two sets that have no elements in common are called **disjoint sets.** If A and B are any two disjoint sets, then $A \cap B = \emptyset$. For example, there are no elements common to both $\{50, 51, 54\}$ and $\{52, 53, 55, 56\}$, so these two sets are disjoint.

$$\{50, 51, 54\} \cap \{52, 53, 55, 56\} = \emptyset$$

EXAMPLE 3 Finding the Intersection of Two Sets

Find each of the following.

(a) $\{9, 15, 25, 36\} \cap \{15, 20, 25, 30, 35\}$

(b) $\{2, 3, 4, 5, 6\} \cap \{1, 2, 3, 4\}$

(c) $\{1, 3, 5\} \cap \{2, 4, 6\}$

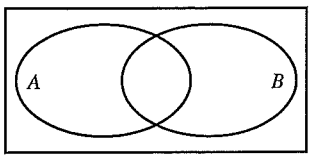
SOLUTION

(a) $\{9, 15, 25, 36\} \cap \{15, 20, 25, 30, 35\} = \{15, 25\}$
The elements 15 and 25 are the only ones belonging to both sets.

(b) $\{2, 3, 4, 5, 6\} \cap \{1, 2, 3, 4\} = \{2, 3, 4\}$

(c) $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$ Disjoint sets

Now Try Exercises 59, 65, and 75.



$A \cup B$

Figure 4

The set of all elements belonging to set A ***or*** to set B (or to both) is called the **union** of the two sets, written $A \cup B$. For example, if $A = \{1, 3, 5\}$ and $B = \{3, 5, 7, 9\}$ then we have the following.

$$A \cup B = \{1, 3, 5\} \cup \{3, 5, 7, 9\} = \{1, 3, 5, 7, 9\}$$

The Venn diagram in **Figure 4** shows two sets A and B. Their union, $A \cup B$, is in color.

EXAMPLE 4 Finding the Union of Two Sets

Find each of the following.

(a) $\{1, 2, 5, 9, 14\} \cup \{1, 3, 4, 8\}$

(b) $\{1, 3, 5, 7\} \cup \{2, 4, 6\}$

(c) $\{1, 3, 5, 7, \ldots\} \cup \{2, 4, 6, \ldots\}$

SOLUTION

(a) Begin by listing the elements of the first set, $\{1, 2, 5, 9, 14\}$. Then include any elements from the second set that are not already listed.

$$\{1, 2, 5, 9, 14\} \cup \{1, 3, 4, 8\} = \{1, 2, 3, 4, 5, 8, 9, 14\}$$

(b) $\{1, 3, 5, 7\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 5, 6, 7\}$

(c) $\{1, 3, 5, 7, \ldots\} \cup \{2, 4, 6, \ldots\} = N$ Natural numbers

Now Try Exercises 73 and 77.

The **set operations** are summarized below.

Set Operations

Let A and B be sets, with universal set U.

The **complement** of set A is the set A' of all elements in the universal set that do *not* belong to set A.

$$A' = \{x \mid x \in U, \quad x \notin A\}$$

The **intersection** of sets A and B, written $A \cap B$, is made up of all the elements belonging to both set A *and* set B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The **union** of sets A and B, written $A \cup B$, is made up of all the elements belonging to set A *or* to set B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

R.1 Exercises

Identify each set as finite *or* infinite. *Then determine whether* 10 *is an element of the set.* ***See Example 1.***

1. $\{4, 5, 6, \ldots, 15\}$

2. $\{1, 2, 3, 4, 5, \ldots, 75\}$

3. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$

4. $\{4, 5, 6, \ldots\}$

5. $\{x \mid x$ is a natural number greater than $11\}$

6. $\{x \mid x$ is a natural number greater than or equal to $10\}$

7. $\{x \mid x$ is a fraction between 1 and $2\}$

8. $\{x \mid x$ is an even natural number$\}$

Use set notation, and list all the elements of each set. ***See Example 2.***

9. $\{12, 13, 14, \ldots, 20\}$

10. $\{8, 9, 10, \ldots, 17\}$

11. $\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}\}$

12. $\{3, 9, 27, \ldots, 729\}$

13. $\{17, 22, 27, \ldots, 47\}$

14. $\{74, 68, 62, \ldots, 38\}$

15. $\{x \mid x$ is a natural number greater than 7 and less than $15\}$

16. $\{x \mid x$ is a natural number not greater than $4\}$

Insert $\in$ *or* $\notin$ *in each blank to make the resulting statement true.* ***See Examples 1 and 2.***

17. 6 _____ $\{3, 4, 5, 6\}$

18. 9 _____ $\{3, 2, 5, 9, 8\}$

19. -4 _____ $\{4, 6, 8, 10\}$

20. -12 _____ $\{3, 5, 12, 14\}$

21. 0 _____ $\{2, 0, 3, 4\}$

22. 0 _____ $\{0, 5, 6, 7, 8, 10\}$

23. $\{3\}$ _____ $\{2, 3, 4, 5\}$

24. $\{5\}$ _____ $\{3, 4, 5, 6, 7\}$

25. $\{0\}$ _____ $\{0, 1, 2, 5\}$

26. $\{2\}$ _____ $\{2, 4, 6, 8\}$

27. 0 _____ $\emptyset$

28. $\emptyset$ _____ $\emptyset$